Let

$$f(w) = \sum_0^n i a_i w^{n-i}, \quad |a_0| > 0,$$

a_i a complex number, be a polynomial in w which does not vanish on a circle S or without it, S having a radius R and the origin for center. Assume first that $f(w)$ has in S a lower limit L; and second that $L>0$. Then there is in S an unending sequence of values,† w,

$$w_1, w_2, \ldots, w_n, \ldots \tag{1}$$

defining a number or point W within or on S, with which can be associated an unending sequence of positive numbers, ϵ,

$$\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots, \epsilon_i \geqq \epsilon_{i+1}, \lim_{i \doteq \infty} \epsilon_i = 0, \tag{2}$$

such that

$$0 < |f(w_i)| - L < \epsilon_i, \quad i=1, 2, \ldots, \infty. \tag{3}$$

Now employing in part Chrystal's notation,

$$f(w+h) = f(w) + \sum_1^n{}^j B_j h^j, \tag{4}$$

where B_j is a polynomial in w of degree $n-j$. $B_n=a_0$, is different from zero, and independent of w. It may be shown, after the manner of Chap. V, §16, that B_j, if it vanish at all, cannot vanish for more than $n-j$ values w. Therefore, there are at most $n(n-1)/2$ values, w, for which some B_j or every B_j vanishes. Some of these values w for which the B's vanish may coincide with W, say τ of them, $0 \leqq \tau \leqq n(n-1)/2$. Since the number of values, or points, w, for which the B's vanish is finite, it is possible to draw from W as a center a circle S' having on or within it only those vanishing points of the B's coincident with the W, τ in all. The circle S' may lie wholly within or partly without S. Henceforth we are concerned only with the region common to the two circles S and S'. The boundary of this common region we designate by S_1.

Let us further assume that for $w=W$,

†We assume here the continuity of $f(w)$ in the neighborhood of the points giving rise to the limit value L. It would be easy and, perhaps, much simpler to prove, once for all, that every rational integral function of w is continuous everywhere in the finite plane; that its absolute value has a lower limit in every bounded region of this plane; that the absolute value of the function actually assumes this limit value for some point within or on the boundary of this region.

$$B_1=B_2=\ldots=B_{m-1}=0, \quad |B_m|>0,\ 0<m\leqq n.$$

Since B_m does not vanish for $w=W$ it does not vanish within or on S_1. Inasmuch as B_j, $j=1, 2, \ldots, n$, is a polynymial in w and $|f(w)|>L$ in S_1, then $B_j/f(w)$ does not become infinite in S_1. Hence there is a positive num- D such that in S_1,

$$\left|\frac{B_j}{f(w)}\right|<D,\ j=1, 2, \ldots, n. \tag{5}$$

In particular, let

$$\left|\frac{B_m}{f(w)}\right|<D_m. \tag{6}$$

Since B_m does not vanish within or on S_1 we assume that within this region $\left|\frac{B_m}{f(w)}\right|$ has a lower limit d_m, $d_m>0$.

From some value of j on, say $j\geqq k$, all terms w_j of the sequence (1) are in S_1. Only the values w_j, $j\geqq k$, will be considered in what follows.

Now in equation (4) replacing w and h by w_l and h_l, we have

$$f(w_l+h_l)=f(w_l)+\sum_1^n{}^j B_{j,l}h_l{}^j, \quad l\geqq k, \tag{7}$$

where the subscript l in $B_{j,l}$ signifies that the B's are now polynomials in w_l. Let us set

$$\frac{B_{j,l}}{f(w_l)}=b_{j,l}(\cos a_{j,l}+i\sin a_{j,l}),$$

$$h_l=r(\cos\theta_l+i\sin\theta_l).$$

Then

$$\frac{f(w_l+h_l)}{f(w_l)}=1+\sum_1^n{}^j b_{j,l}r^j\Theta_{j,l}, \tag{8}$$

where

$$\Theta_{j,l}=\cos(j\theta_l+a_{j,l})+i\sin(j\theta_l+a_{j,l}).$$

We now assign to θ_l values such that

$$\Theta_{m,l}=-1, \quad l\geqq k. \tag{9}$$

Equation (8) becomes, consequently,

(10) $$\frac{f(w_l+h_l)}{f(w_l)}=1-b_{m,\,l}r^m+\sum_1^n{}^j b_{j,\,l}\,r^j\,\Theta'_{j,\,l},\quad l\geqq k,\ j\neq m,$$

where the accent on $\Theta'_{j,\,l}$ signifies that θ_l now has the value assigned by equation (9). Hence

(11) $$1-b_{m,\,l}-\left|\sum_1^n{}^j b_{j,\,l}\,r^j\Theta'_{j,\,l}\right|\leqq\left|\frac{f(w_l+h_l)}{f(w_l)}\right|\leqq 1-b_{m,\,l}\,r^m$$
$$+\left|\sum_1^n{}^j b_{j,\,l}\,r^j\,\Theta'_{j,\,l}\right|,\quad l\geqq k,\ j\neq m.$$

By equation (6) and the remark following,

(12) $$d_m\leqq b_{m,\,l}<D_m;\ \text{further,}$$

(13) $$\left|\sum_{m+1}^n{}^j b_{j,\,l}\,r^j\Theta'_{j,\,l}\right|\leqq\sum_{m+1}^n{}^j b_{j,\,l}\,r^j<D\sum_{m+1}^n{}^j r^j.$$

For brevity, let

(14) $$\phi_l=\left|\sum_1^{m-1}{}^j b_{j,\,l}\,r^j\Theta'_{j,\,l}\right|.$$

We now take r, $0<r<1$, so that

$$r^{m+1}>r^{m+2}>\ldots>r^n.$$

Hence,

(15) $$(n-m)r^{m+1}>\sum_{m+1}^n{}^j r^j.$$

The inequalities (12) may now be replaced by

(16) $$1-D_mr^m-\phi_l-(n-m)Dr^{m+1}<\left|\frac{f(w_l+h_l)}{f(w_l)}\right|$$
$$<1-d_mr^m+\phi_l+(n-m)Dr^{m+1}.$$

We may now choose r_1, $r_1>0$, so that, $r<r_1$,

(17) $$1-D_mr^m-(n-m)Dr^{m+1}>2g,\ g>0,\ \text{and a constant;}$$

and likewise, r_2, $r_2>0$, so that, $r<r_2$,

(18) $$1-d_mr^m+(n-m)Dr^{m+1}<1-2f,\ f>0,\ \text{and a constant.}$$

Hence there is a number r_3, $0<r_3<r_1$, $r_3<r_2$, such that, $r \leqq r_3$, the inequalities (17) and (18) are both satisfied.

Let it be assumed that $b_{j,l}$, $j=1, 2, \ldots, (m-1)$, continually approaches zero when w_l approaches W as a limit. Hence we can choose an integer k_1 so that $l>k_1$,

$$(19) \qquad | b_{j,l}\, \Theta'_{j,l} | \leqq f'/m,\ j=1, 2, \ldots, m-1,\ 0 \leqq f' < f,\ f' < g,$$

where f' is a constant. Hence

$$(20) \qquad \phi_l \leqq f',\ l>k_1.$$

Therefore, by (17), (18), and (20),

$$(21) \qquad g < \left| \frac{f(w_l+h_l)}{f(w_l)} \right| < 1-f,\ l>k_1,\ r \leqq r_3.$$

Let

$$\left| \frac{f(w_l+h_l)}{f(w_l)} \right| = 1-\mu_l.$$

Then

$$(22) \qquad g<1-\mu_l<1-f,\ l>k_1,\ r \leqq r_3,$$

whence

$$(23) \qquad 1-g>\mu_l>f.$$

Therefore, μ_l is a positive proper fraction which can approach neither zero nor one as a limit when l approaches infinity. Hence

$$(24) \qquad \begin{aligned} | f(w_l+h_l) | &= (1-\mu_l)\, | f(w_l) | = (1-\mu_l)(L+\epsilon) \\ &= L+\varepsilon-\mu_l(L+\epsilon),\ l+k_1,\ 0 \leqq r \leqq r_3. \end{aligned}$$

Now, there is an integer k_2 such that for $l>k_2$ and for r, $0<\delta \leqq r \leqq r_3$, where $\delta<r_3$ and is a constant,

$$(25) \qquad L+\varepsilon-\mu(L+\varepsilon)<L.$$

Let us now choose a number r_4, $r_4<r_3$, so that

$$(26) \qquad 0<r_4<R-|w_l|,\ k_2<l<k_3,\ k_3 \text{ an integer},$$

and further condition δ by the inequality, $\delta<r_4$. Then the inequality (25) holds for every point w_l+h_l such that $k_2<l<k_3$ and $\delta<r<r_4$; further, every such point is in S.

Therefore, L, $L>0$, is not the lower limit of $f(w)$ in S. Hence $L=0$, and $f(w)$ has a root in S.

THE PERFECT MAGIC SQUARES FOR ∓1909.*

By THEODORE L. DeLAND, Treasury Department, Washington, D. C.

In an article on Magic Squares in *Our Schoolday Visitor, Mathematical Almanac and Annual*, 1871, Judge Scott of the Ohio Supreme Court divides them into, simple, perfect, complex, compound, triple, quadruple and quintuple magic squares. I will add one more — potential magic squares, or those existing in possibility, not in reality — being too great to construct.

There can be magic squares — and some very interesting ones — that involve both positive and negative numbers, that respond to general laws for their development. The magic squares for the years ∓1909 all belong to this class.

Let y=the year; then we have by Zerr's formula†,

$$2y=n[2x\pm(n^2-1)];$$

from which

$$x=[2y\mp(n^3-n)]/2n.$$

We know that $1909=23\times83$, the product of two primes, from which

I. When $y=1909$, and $n=23$, we have for the least value, $x=-181$; and for the greatest value, $x=347$.

II. When $y=1909$, and $n=83$, we have for the least value, $x=-3421$; and for the greatest value, $x=3457$.

III. When $y=-1909$, and $n=23$, we have for the least value, $x=-347$; and for the greatest value, $x=181$.

IV. When $y=-1909$, and $n=83$, we have for the least value, $x=-3467$; and for the greatest value, $x=3421$.

Judge Scott also claims that if n be a prime number, not less than 5, we may in every such case construct a great variety of perfect magic squares, and gives the general rule for their construction; and concludes with the remark, that the number of such magic squares may be expressed by the formula, $n^2(n-1)^2(n-3)(n-4)$.

The number of systems of magic squares for the years, −1909 and +1909 is limited to I, II, III, and IV, given above, and to the variations of each.

By the formula we have for

I. 97,293,680 magic squares for 1909;

II. 292,752,739,520 magic squares for 1909;

III. 97,293,680 magic squares for −1909; and

IV. 292,752,739,520 magic squares for −1909; or a

Total of 585,700,066,400 magic squares for the year +1909 and −1909.

*In the January, 1909, issue of the MONTHLY, magic squares for 1909 were requested. This article is in response to that request.

†Vol. XVI, No. 1, page 2, of the MONTHLY.

We will now develop a perfect magic square for -1909, when $n=23$. The year, -1909, may be interpreted as before Christ. In each series from $-x$ to $+x$, zero must occupy one cell in all of these magic squares.

There is no magic square for positive integers alone for 1909.

The following perfect magic square for -1909 begins with -347 and ends with $+181$; but the corresponding magic square for $+1909$ would begin with -181 and end with $+347$.

MAGIC SQUARE FOR THE YEAR, -1909.

−70	−46	−20	8	30	56	80	105	130	155	180	−347	−322	−297	−272	−247	−222	−197	−172	−147	−122	−97	−72
−46	−21	4	29	54	79	104	129	154	179	−325	−323	−298	−273	−248	−223	−198	−173	−148	−123	−98	−73	−71
−22	3	28	52	78	103	128	153	178	−326	−324	−299	−274	−249	−224	−199	−174	−149	−124	−99	−74	−49	−47
2	27	52	77	102	127	152	177	−327	−302	−300	−275	−250	−225	−200	−175	−150	−125	−100	−75	−50	−46	−23
26	51	76	101	126	151	176	−328	−303	−301	−276	−251	−226	−201	−176	−151	−126	−101	−76	−51	−26	−24	1
50	75	100	125	150	175	−329	−304	−278	−277	−252	−227	−202	−177	−152	−127	−102	−77	−52	−27	−29	0	25
74	99	124	149	174	−330	−305	−280	−278	−253	−228	−203	−178	−153	−128	−103	−78	−53	−28	−3	−1	24	49
98	123	148	173	−331	−306	−281	−256	−254	−229	−204	−179	−154	−129	−104	−79	−54	−29	−4	−2	23	48	73
122	147	172	−332	−307	−282	−257	−255	−230	−205	−180	−155	−130	−105	−80	−55	−30	−5	20	22	47	72	97
146	171	−333	−308	−283	−258	−233	−231	−206	−181	−156	−131	−106	−81	−56	−31	−6	19	21	46	71	96	121
170	−334	−309	−284	−259	−234	−232	−207	−182	−157	−132	−107	−82	−57	−32	−7	18	43	45	70	95	120	145
−335	−310	−285	−260	−235	−210	−208	−183	−158	−133	−108	−83	−58	−33	−8	17	42	44	69	94	119	144	169
−311	−286	−261	−236	−211	−209	−184	−159	−134	−109	−84	−59	−34	−9	16	41	66	68	93	118	143	168	−336
−287	−262	−237	−212	−187	−185	−160	−135	−110	−85	−60	−35	−10	15	40	65	67	92	117	142	167	−337	−312
−263	−238	−213	−188	−186	−161	−136	−111	−86	−61	−36	−11	14	39	64	89	91	116	141	166	−338	−313	−288
−239	−214	−189	−164	−162	−137	−112	−87	−62	−37	−12	13	38	63	88	90	115	140	165	−339	−314	−289	−264
−215	−190	−165	−163	−138	−113	−88	−63	−38	−13	12	37	62	87	112	114	139	164	−340	−315	−290	−265	−240
−191	−166	−141	−139	−114	−89	−64	−39	−14	11	36	61	86	111	113	138	163	−341	−316	−291	−266	−241	−216
−167	−142	−140	−115	−90	−65	−40	−15	10	35	60	85	110	135	137	162	−342	−317	−292	−267	−242	−217	−192
−143	−118	−116	−91	−66	−41	−16	9	34	59	84	109	134	136	161	−343	−318	−293	−268	−243	−218	−193	−168
−119	−117	−92	−67	−42	−17	8	33	58	83	108	133	158	160	−344	−319	−294	−269	−244	−219	−194	−169	−144
−95	−93	−68	−43	−18	7	32	57	82	107	132	157	159	−345	−320	−295	−270	−245	−220	−195	−170	−145	−120
−94	−69	−44	−19	6	31	56	81	106	131	156	181	−346	−321	−296	−271	−246	−221	−196	−171	−146	−121	−96

ON THE TEACHING OF ANALYTIC GEOMETRY.

By A. E. YOUNG, Miama University.

The idea of the graph, the basal concept in the development of that branch of mathematics known today as analytic geometry, was a natural outgrowth, it would seem, of the line of thought undertaken by the great philosopher and mathematician Descartes; for he undertook to parallel the algebraic operations with geomatric, and was probably the first to obtain the roots of a quadratic equation in one unknown by aid of ruler and compasses. The next step, the solution of a quadratic in two unknowns, geometrically interpreted, as in the previous case, led to the coordinate idea, and hence to the graph.

Thus was born the concept which has made possible the science of mathematics as it exists today. Algebra and geometry were thus united by Descartes and others, and since their time analysis proper and geometry have continued their development hand in hand. In fact, analytic geometry may be defined as the theory of analysis geometrically interpreted. The former is therefore a dependent science, and many mathematicians thus consider it. However, for the better understanding of analysis proper, and its applications to problems of the natural sciences, it has played an important and, oftentimes, an essential role. The sciences of astronomy, physics, and mechanics are what they are today through the theory of analysis geometrically interpreted. The engineering profession is dependent upon the ideas of analytic geometry for the applications of algebra, trigonometry, calculus, differential equations, etc., to the solution of problems. The graph is as useful to the engineer as his foot-rule, and almost as constantly used in many cases. The fundamental idea of analytic geometry, that of showing graphically the relation between two (or more) variables, the change in one and the corresponding change in the other, has crept into all professions and occupations. Many lines of work are almost entirely dependent for their recent growth upon the employment of these ideas. Students return to our technical schools at the close of every summer from factories and mills with problems of loci, etc., which uneducated mechanics have given them for solution. Many men today find themselves seriously handicapped because of their lack of knowledge of this kind and the steadily increasing demand for it. This cry of the industrial world for the solution of difficult problems can not be met in most cases without the aid of analytic geometry.

In view of the foregoing conditions, therefore, we do well to consider the question of the proper method of teaching so important a subject. What should we teach and how should we teach it, are questions which may well be asked, and the answers will vary according to the experience of different teachers.

The writer has found that analytic geometry is most difficult

for the beginning student. This difficulty is in part inherent in the subject, but to a great extent also it is due to the student's entire lack of previous training in this line of work and the failure of the teacher to appreciate this fact. This difficulty is being overcome somewhat, at present, by the introduction of graphical work in high school algebras, which is certainly a move in the right direction. It is to be hoped that in the near future coordinate geometry in its simpler forms may become a preparatory school subject. Of the year and a half now spent in the study of elementary algebra, one half year might well be given to the geometric interpretation of algebra, or, in other words, to the study of coordinates, graphs, simple loci problems, etc. This is desirable not only in order that the pupil may have the coordinate idea well grounded before he comes to college and goes into more advanced work, but also that the pupil who ends his course in the preparatory school, may have an opportunity to learn at least the elementary ideas of a subject which is of such general application in the everyday problems of life.

Given a class of students who have been grounded in the ideas of coordinate geometry, and granting that they have average intelligence, no other branch of mathematics is likely to be so interesting to both teacher and pupils. The work need never be dry to either. The problems are endless and can be graded so as to come within the reach of all. The student can see almost daily improvement in his own ability to grasp the ideas and solve the problems. A new field is open to him, whose vastness is revealed by a few suggestions from the teacher. No subject is so likely as this to turn the young, inexperienced student of mathematics into an enthusiast.

Before discussing the question of the subject matter, or the manner of presenting it, it is necessary to consider the class of students to be taught, why they are studying the subject, and what use they are to make of it. Let us divide them into two classes, the students of the technical school, and those of the college proper. The latter class may again be divided into those who elect the subject and those who take it as required work. In all colleges where mathematics is still a required subject, and there are a few outside of the technical schools, analytic geometry is usually required. Of course this is as it should be, though it may be doubted whether sufficient work of this nature is required in most colleges at the present time. A half year's work four times per week should be the lower limit, and even then the course might well be called "Coordinate Geometry," instead of "Analytic," leaving the latter name for a second course to be described later.

In a course of this kind, given to a class of students many of whom do no more work than they are compelled to do, the teacher meets with a hard problem, but one which is well worth solving, and probably no more difficult than that of teaching any other required subject in these days of electives. The difficulty of teaching required work at the present time is due largely to the ease with which a student can meet the requirements of

the elective undergraduate work in other lines. Analytical geometry is not an easy subject compared to most of the electives. In order to make the work easier for the average student, and that the best may grasp the ideas more clearly, the teacher should spend a large part of the class-period, especially near the beginning of the course, in giving explanations. Loafing on the part of students is easily prevented by requiring them to reproduce the work done by the teacher either at the close of the recitation or at the next meeting of the class, and by frequent exercises.

In regard to the material for a class taking a short course in required work, there should be little question. Cartesian coordinates should give the first introduction to the subject, but polar coordinates should be by no means omitted. The latter are quite as much employed in some lines of practical work as the former, especially in the engineering profession.

Change of coordinates and lines of reference should be discussed till the processes become familiar to the students. Loci problems of the simpler kind should be studied carefully. These problems give great difficulty to the average student, due probably, to the awkward manner in which they are presented, in many cases. If the parametric form of the equation of a curve is considered carefully in the first place, in the case of known curves, then loci problems are easily solved by getting the equation of the curve in this form and stopping there. The elimination of the parameter (or the parameters) is an algebraic question, and one which the student should be taught to consider by itself. It is the failure to distinguish between the two steps that makes the problems difficult.

It would seem that, in general, too much attention in the past has been paid, in a course like the one under discussion, to the consideration of conics and their properties. Oftentimes the dull pupils in the class, after having finally grasped the fact that there are other loci besides straight lines, at once conclude that there must of necessity be conic sections. It would be well to introduce some of the simpler curves of the third and fourth degree, in polar coordinates, as well as cartesian, and also, some of the more common non-algebraic curves such as the catenary, spirals, and cycloids, the latter in the parametric form.

No great results can be hoped for from a course of this kind given under such conditions. If the class as a whole learn thoroughly the common facts, and are at the same time aware that they have just touched the subject itself, and the really good students have the ambition to know more about it, the teacher should be well satisfied, and doubtless would be. That this latter point may be gained it would be well for the teacher throughout the course to generalize for the benefit of the better students whenever the opportunity offers.

For example, from the polar line one may go to the polar curve, giving as a particular case the polar curves corresponding to certain cubics. The ideas of class and order may be considered just enough to arouse the

curiosity of the pupils. The general method for determining asymptotes to an algebraic curve should be explained, and particular examples given. In this way we show the student some of the paths leading out, and he is naturally interested in the question as to where they lead. His ambition to know is stimulated. He begins to see how vast is the science with which he is dealing. He avoids closing the course with the idea that there were just a few points which he did not have time to learn, and he really understands that if he is to know much about the subject he must elect the course to which we now turn our attention, "elective analytic geometry," a course which should be taken up at the beginning of his Sophomore year and carried through two semesters.

We are not proposing here a course for graduate students in mathematics, but a course suited to Sophomores. In our wild attempts these days to do graduate work, we inject too much of the graduate spirit in the courses in mathematics primarily intended for the undergraduate of limited experience. The result is that the undergraduate suffers. He finds himself beyond his depth, and if he ever does touch ground before the close of the course, it is in a half drowned condition. We shall make no such mistake in outlining this course. It is taken for granted that the students who elect it have little mathematical knowledge. At most they have, or are at the same time obtaining, a little knowledge in differential calculus. To obtain the best results this should be the case.

This course should begin with a rapid review of the principal subjects considered in the previous course, namely, the straight line, the properties of conics, the transformation of coordinates, polars and tangents, etc. In connection with the study of the straight line, the idea of the point-line quality should be considered carefully and to a considerable extent. Also, line coordinates and homogeneous point coordinates should be used at the same stage of the work, and thereafter, whenever it is of advantage to do so. In connection with the idea of the tangent, the asymptote should be considered, the coordinates in use being polar as well as rectangular. The idea of the polar curve, first, second, third polar, etc., for algebraic curves should be discussed quite fully, that is, long enough to fix the ideas in the mind of the student. The cubic curves should be studied at some length. They might be divided into the five classes, cuspidal, bipartite, etc., and then the various curves in each class, characterized by the way in which they are cut by the line at infinity. No time should be spent in the careful plotting of unknown curves. Such work should be relegated to a course in calculus.

The course as outlined so far, should cover the first half of the Sophomore year and should fit the student taking it for a more advanced course in analytic geometry, which is generally called Modern Analytic Geometry. This latter course, intended for graduate students or those undergraduates who are specializing, we shall not attempt to outline here.

The great mistake, at the present time, is that of omitting a course similar to the one just outlined, and substituting in its place the one in Modern Analytic Geometry. Having taken the first, the student is infinitely better prepared for the second.

Before outlining the work of the second half year of this course, which should be in geometry of three dimensions, let us say a few words in regard to the work in analytic geometry at colleges where no mathematical work is required. Here the course should be arranged somewhat differently, we believe. The first course in analytic geometry should be made more comprehensive than when the course is required. It should include all of the work of the required course as outlined above, and some of the subjects just touched upon there should be developed more fully. A half year, four times per week, would be necessary for this work. It should be followed directly by the course in geometry of three dimensions which we are about to consider, modified somewhat to meet the new conditions.

The discussion of geometry of three dimensions should begin, of course, with a careful study of the straight line and plane, followed by work on the transformation of coordinates, change of origin and axes, etc. The discussion of the conicoids should be thorough. The general equation of the second degree should be discussed, as should also the subject of confocal conicoids. Polar and spherical coordinates should be employed in the discussion of some surfaces in which they can be used to advantage, With students who have had the required work as well as the elective course in plane analytics, the work should be done rapidly. It should be followed by an elementary discussion of space curves and algebraic surfaces. The one and two parameter idea, corresponding respectively to the curve and surface, should be touched upon sufficiently to fix the fundamental thought. The work in discussing algebraic surfaces would parallel the similar discussion of plane algebraic curves, and yield large results with little work. A class composed of students who have taken only the one course (the elective) would hardly be able to do thoroughly more than the work first outlined above.

Turning now to the teaching of analytical geometry in the Technical School, we have quite a different case for consideration. Here the classes are composed of students, the majority of whom, at least, realize that they are taking, in analytical geometry, one of the subjects which will prove most useful in their actual engineering work, and one the mastery of which is essential to success in several of the subjects which are to follow. Moreover, we can assume (though we find so many exceptions that we begin to doubt the rule) that the students have a little more mathematical knowledge when beginning the subject of analytic geometry than in the previous case.

External conditions are somewhat different from those in an academic institution. The engineering professors are demanding that the student in the technical school absorb all the mathematics that he is to have in his

first year and a half, or in the first two years at most. He thus has at best little time to assimilate his mathematical food, and we do not wonder that most of it leaves only a fleeting impression. In this connection we call attention to the books which are being written purposely to meet these conditions. We refer to books on "mathematics for engineers" in which courses like calculus, analytic geometry, etc., are combined. We have no doubt but that such a condition of affairs is bad for the student with the average preparation in mathematics.

It is two much on the face of it to expect that a freshmen, prepared as ours are at present, will be able to take up a course in "engineering mathematics," in which are combined the ideas of half a dozen subjects of which he knows little or nothing, and obtain any very definite knowledge at the end of the course. "Engineering mathematics" should *follow* thorough courses in the purely mathematical subjects and should not *replace* these. Taken thus it would prove of immense benefit to the students. Taken as it is, the principles of mathematics which are so important, and a knowledge of which is so essential to the student, are in most cases not clearly understood, if grasped at all.

We do believe, however, that a quite different course in analytic geometry should be given the prospective engineer from that provided for the ordinary student. In his case it should be always kept in mind that this is a subject which he will doubtless use in a practical way ever afterward. The difficult problem which the engineer will often meet is this: Given certain conditions, what is the equation of a curve, or curves, corresponding? Or this: Given certain corresponding values of two variables represented, say by two points in a plane, to find a relation between the variables which will approximate the true relation sufficiently near for his purpose? In the writer's short experience as a teacher in an engineering school, several graduates have reported difficulty in regard to such questions. Their great mistake seemed to be their assumption that to a set of points there corresponds necessarily one curve and only one. They had not fully grasped the facts necessary to the full understanding of loci problems, nor was it to be wondered at considering the circumstances.

As of first importance, therefore, let us urge the claims of graphical work, and the great care that should be taken by the instructor to explain most carefully the theory underlying the determination of a curve which will satisfy given conditions. Many problems of the following nature should be treated: Given a certain number of points, determine whether certain algebraic curves such as the circle, the parabola, the ellipse, and certain third degree curves, pass through them. Take up cases where one arbitrary constant remains in the equation, impose another condition, etc,, etc. Consider the determination of loci whose equations hall be of some of the more common non-algebraic forms, as the catenary, spirals, or the trochoids. Consider the form of the equations of some of these if the origin and axes

are changed. Be careful to see that the pupils have a great familiarity with the ideas involved in the transformation of coordinates.

The next most important part of their work should be the plotting of curves from given equations. A student who left college at the close of his junior year, reported after being out three years, engaged in construction work for one of our large railroad companies, that the study of "higher plane curves," had proved of the greatest use to him of any of his mathematical work. The ideas gained enable him to treat his problems intelligently. This sort of work, however, cannot be done without using the ideas and methods of the calculus. Partly for that reason, and partly in order that the student may begin early the study of this great subject, we would urge that no extended course be taken in analytical geometry, apart from calculus. A semester's work should enable the student to grasp the fundamental ideas of the subject of plane and solid analytic geometry. The course should then continue, but as a course in differential calculus, geometrically applied. It is a mistake to fill a book on elementary calculus with problems from all the sciences in which it may be applied. Much better would it be to confine the applications almost entirely to the science of elementary analytical geometry. In this way the student will stand a much better chance of actually grasping the principles of the calculus and will at the same time learn a great deal, and in the proper way, about a very necessary subject. It may be that when the student takes up mechanics and allied subjects, he will not start off so briskly at first but in the end he should do much better work, as he should have the principles of calculus in better shape and hence be able to apply them better. An engineer of long experience, in stating that he never made any use of calculus in his work, seemed to think that it was due to the way in which the subject had been presented to him. No doubt he was right about it, although we might disagree as to the way it should be presented. An intelligent, well-educated engineer *should* use calculus to his advantage in some of his work. In fact, the gentleman referred to above did say that engineers in the United States Government service do use it continually. A man can not use calculus understandingly unless he is familiar with the *theory* of calculus. A careful study of the theory should precede its general application, and to the fact that this is not the case in many of our technical schools, we owe the "failure of the science" as an engineering tool. It would seem that today the mathematicians are falling into this very error themselves, urged on by the professors in the applied sciences. They say to themselves, "I must not teach any pure mathematics," and they almost apologize to their classes for wasting any time on calculus itself, or analytic geometry before applying it. The result is a great deal like that of beginning the study of Greek literature before learning the Greek language. They make a mistake in this, and some of our leading educators in large engineering schools are becoming convinced of this fact. They say, "Stick to your own science. Give the

students the pure mathematical ideas, and let the professors of physics, mechanics, etc., do the applying of the principles." Much better would it be to give the non-engineering student work in the broader applications of the mathematical theory, for the engineer is sure to get this work, and the other students will have no other chance.

Finally, in a course in analytic geometry for engineers, we would urge again the general use of the parametric form of the equation. It has a wide application and is of common use in their profession, as, for example, in the case of the moving point whose coordidates are functions of the time. Polar coordinates, also, are almost as familiar to the engineer in much of his work as rectangular, and should receive careful attention in the course in analytic geometry, both before and after the application of the theory of calculus.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

323. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

Show that the relation $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$ can hold for real numbers only when they are in proportion.

Solution by the PROPOSER.

Expanding and rearranging, we get

$$a^2(c^2+d^2)-2abc(b+d)+b^4+b^2c^2-2bc^2d+b^2d^2+c^4=0.$$

Solving for a,

$$a=\frac{bc(b+d)\pm(c^2-bd)\sqrt{[-(b^2+c^2+d^2)]}}{c^2+d^2}$$

which can be real only when $c^2-bd=0$, *i. e.* when $c/d=b/c$. Then

$$a=\frac{bc\ (b+d)}{c^2+d^2}=\frac{bc\ (b+d)}{bd+d^2}=\frac{bc}{d},$$

since $b+d\neq 0$. Therefore, $a=\frac{bc}{d}$ or $\frac{a}{b}=\frac{c}{d}=\frac{b}{c}$.

Also solved by G. B. M. Zerr.

324. Proposed by R. D. CARMICHAEL, Princeton University.

Sum the *finite* series

$$\frac{16n^2-2^2}{4!}-\frac{(16n^2-2^2)(16n^2-4^2)}{6!}+\frac{(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}-\ldots$$

where n is a positive integer.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

$$\cos m\theta=1-\frac{m^2}{2!}\sin^2\theta+\frac{m^2(m^2-2^2)}{4!}\sin^4\theta-\frac{m^2(m^2-2^2)(m^2-4^2)}{6!}\sin^6\theta+\ldots$$

Let $m=4n$, then

$$\cos 4n\theta=1-\frac{16n^2}{2!}\sin^2\theta+\frac{16n^2(16n^2-2^2)}{4!}\sin^4\theta$$

$$-\frac{16n^2(16n^2-2^2)(16n^2-4^2)}{6!}\sin^6\theta+\ldots$$

Let $\theta=\frac{1}{2}\pi$, then $\cos 4n\theta=\cos 2n\pi=1$.

$$\therefore 1=1-\frac{16n^2}{2!}+\frac{16n^2(16n^2-2^2)}{4!}-\frac{16n^2(16n^2-2^2)(16n^2-4^2)}{6!}$$

$$+\frac{16n^2(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}-\ldots$$

$$\therefore \tfrac{1}{2}=\frac{(16n^2-2^2)}{4!}-\frac{(16n^2-2^2)(16n^2-4^2)}{6!}$$

$$+\frac{(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}+\ldots$$

GEOMETRY.

345. Proposed by LLOYD HOLSINGER, Bradley Polytechnic Institute, Peoria, Ill.

If a variable polygon move in such a way that its n sides turn severally round n fixed points O_1, O_2, ..., O_n while $n-1$ of its vertices slide, respectively, along $n-1$ fixed straight lines v_1, v_2, ..., v_{n-1}, then the last vertex will describe a conic; and the locus of the point of intersection of any pair of non-adjacent sides will also be a conic. Cremona's *Projective Geometry*.

Solution by HOWARD C. FEEMSTER, A. B., York College, York, Neb.

Let the sides of the polygon $a_1, a_2, \ldots, a_n$ turn severally around the n fixed points $O_1, O_2, \ldots, O_n$, describing n flat pencils, $O_1, O_2, \ldots, O_n$, while the $n-1$ vertices $A_1, A_2, \ldots, A_{n-1}$ slide, respectively, along the $n-1$ fixed straight lines, $v_1, v_2, \ldots, v_{n-1}$, forming $n-1$ ranges, $A_1, A_2, \ldots, A_{n-1}$. Pencils O_1 and O_2 are perspective with range A_1; ranges A_1 and A_2 are perspective with pencil O_2; pencils O_2 and O_3 are perspective with range A_2, thus making the n pencils and $n-1$ ranges projective.

Now the projective pencils O_1 and O_n or any two of the non-consecutive pencils (using above order) will not, in general, be perspective, and every such intersection will thus describe a conic range as its locus. *Encyclopedia Britannica*, "Projective Geometry," Sec. 45.

347. Proposed by W. J. GREENSTREET, M. A. Marling School, Stroud, England.

ABC is a triangle, and *D*, *E*, *F*, are the mid points of the arcs of its nine-point circle cut off by *BC*, *CA*, *AB*, respectively. The inscribed circle touches these sides at *X*, *Y*, *Z*. Are the lines *DX*, *EY*, *FZ* concurrent? A purely geomatrical discussion required.

Solution by BENJ. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

In my *Mathematical Solution Book*, page 461, it is shown that the perpendiculars to the chords cut off by the nine-point circle, at the mid-points intersect in a point *S*, which is the center of the nine-point circle. These perpendiculars produced from the chords to the subtended arcs, bisect the smaller arcs in *D*, *E*, and *F*. Since the in-circle is tangent to the chords or chords produced in *X*, *Y*, *Z*, perpendiculars erected at these points meet in a point, *I*, the center of the in-circle. Drawing the lines *DX* and *EY*, and producing them until they meet *IS*, or *IS* produced, in *C* and *C'*, respectively, we have the triangles *SDC* and *SDC'*.

The triangles *SDC* and *IXC* are similar, as are also *SDC'* and *IYC'*. Hence, *SD*, (the radius of the nine-point circle): *IX*, (the radius of the in-circle) $=SC : IC$; and similarly, *SE*, the radius of the nine-point circle: *IY*, the radius of the in-circle, $=SC' : IC'$. Hence, $SD-IX : IX=SC-IC(=SI) : IC$, and $SE-IY : IY=SC'-IC'(=SI) : IC'$. Hence, $IC=IC'$ and, therefore, *DX* and *EY* intersect the line joining the nine-point circle and the in-circle in the common point *C*. In the same way, it may be shown that the line *FZ* intersects the same line in the point *C*. Hence, the three lines are concurrent.

CALCULUS.

275. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Explain fully why the circular measure of an angle is used in the calculus.

Solution by the PROPOSER.

There is one good reason for changing from the *gradual measure* to the *circular measure* in the calculus.

In finding the derivative of $\sin u$ we are required to evaluate

$$\lim_{u \doteq 0} \left| \frac{\sin \frac{1}{2}u}{\frac{1}{2}u} \right|,$$

which becomes equal to unity.

Had we used degrees, we would have introduced the extraneous factor $\pi/360$. For, suppose u to be given in degrees, minutes or seconds; then we have

$$\frac{d}{dx}(\sin u^\circ) = \frac{\pi}{180} \cos u^\circ \frac{du}{dx}; \quad \frac{d}{dx}(\sin x') = \frac{\pi}{180.60} \cos x';$$

$$\frac{d}{dx}(\sin x'') = \frac{\pi}{180.60.60} \cos x''.$$

Taking a numerical example, let $x = 20^\circ$ and $\triangle x = .0001^\circ$.

From a table of natural sines, radius being 1, we obtain $\sin 20^\circ = .3420201 = u$, say; and, $\sin 20.0001^\circ = .34202174 = u + \triangle u$. $\therefore \triangle u = .00000164$.

Now, the length of a circular arc subtending an angle of $.0001^\circ$ is, (radius being 1) $.0000017453 = \triangle x$.

$$\therefore \frac{\triangle u}{\triangle x} = \frac{\text{increment of sine}}{\text{increment of arc}} = \frac{.00000164}{.0000017453} = .93967,$$

which is the value of $\cos 20^\circ$ correct to four places. The actual value, found by indefinitely decreasing the increment, is

$$\frac{du}{dx} = \cos 20^\circ = .9396926.$$

Remark by BENJAMIN F. FINKEL.

An angle, being a measurable magnitude, must be measured, or compared, by some arbitrarily chosen magnitude of its own kind. This arbitrarily chosen angular magnitude, we call the *unit angle*. In geometry, the unit angle is the angle formed by two straight lines perpendicular to each other. This unit is too large for trigonometric purposes, and so a smaller angle is chosen as a unit, namely, the 90th part of a right angle, called a degree. Any other arbitrarily chosen part would serve analysis as well, though the convenience or inconvenience in computation would be modified

thereby. Now in elementary geometry it is proved that angles at the center are proportional to the intercepted arcs.

Let $s[L]$, $r[L]$, and θ be the arc, radius, and center angle, s and r being measured in the same *length-unit*, the angle unit being at present undetermined. Then $s[L]=kr[L]\theta$. Let us now *assume* $\theta=1$, *the unit angle, when* $s=r$. Hence, $k=1$, under this assumption. We thus have as our unit angle, $[\Theta]$, the angle at the center of a circle, which intercepts an arc equal in length to the radius of the circle. Hence, we have the general relation $s[L]=r[L]\,\theta\,[\Theta]$ and $\theta[\Theta]=s/r$. This unit angle, $[\Theta]$, is called a *radian*. In terms of the fundamental units of length, mass, and time, it is of zero dimensions, since $[\Theta]=\frac{s[L]}{r[L]}[M^\circ][T^\circ]=L^\circ M^\circ T^\circ$, s being equal to r. From this we see that the great advantage of this unit of angular measure over that of any other is that it avoids, as remarked by Mr. Schmall, the introduction of a *coefficient of variation* different from unity.

PROBLEMS FOR SOLUTION.

ALGEBRA.

329. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

Between the quantities a and b there are inserted n arithmetical and n harmonical means, and a series of n terms is formed by dividing each arithmetical by the corresponding harmonical mean. Show that the sum of the series is, $n\left[1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right]$.

330. Proposed by R. D. CARMICHAEL, Princeton, N. J.

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n=\infty} x^{1/(2n-1)},\ n=1,\ 2,\ \ldots;\quad f(x)=\lim_{n=\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

GEOMETRY.

357. Proposed by E. R. HOYT, St. Louis, Mo.

A room is 30 feet long, 12 feet wide, and 12 feet high. At one end of the room, 3 feet from the floor, and midway from the sides, is a spider. At the other end, 9 feet from the floor, and midway from the sides, is a fly. Determine the shortest path by way of the floor, ends, sides, and ceiling, the spider can take to capture the fly.

358. Proposed by H. C. FEEMSTER, A. B., Professor of Mathematics, York College, York, Neb.

Cut four coplanar non-copunctual straight lines in a harmonic range.

CALCULUS.

387. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

An object P, being placed beyond the principal focus F of a convex lense, determine its position when its distance PQ, from its image Q, is a minimum.

388. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

Find $\int \frac{xdx}{(1+x^3)^{\frac{2}{3}}}$.

MECHANICS.

240. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A simple beam length $2a$, supported at both ends, is loaded in the form of a parabola, height of vertex b. Find deflection at center due to this load.

241. Proposed by C. N. SCHMALL, 604 East 5th Street, New York City.

In a certain New York theatre there is an asbestos curtain supported by thin circular rings, radius r, which move on a cylindrical rod of radius a. The curtain is intended to be drawn by a *steady pull*. Taking μ as the coefficient of friction, show that this will not be possible if r be less than $a\sqrt{(1+\mu^2)}$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

170. Proposed by PATRICK WALSH, 1451 Annunciation Street, New Orleans, La.

The areas of rectangles A and B are respectively 15170 10/27 and 31230.3627. Find the sides and diagonal of each rectangle in exact or rational numbers.

NOTES AND NEWS.

The editors extend to all our readers the Greetings of the Coming Year. We hope that delays in the regular appearance of THE MONTHLY will not occur during 1910. We trust that our subscribers will continue their support and that we may have the immediate renewals of those who have not already remitted their subscriptions for the new year.

On December 23, Editor Miller was married to Miss Cassie A. Boggs. They will make their home in Urbana, Illinois. The other editors and all readers of the monthly extend congratulations to the happy couple.

Dr. G. E. Fisher, of the University of Pennsylvania, has been appointed Dean of the College of Arts and Sciences, and Dr. I. J. Schwatt, of the same University, has been promoted to full professorship of mathematics. F.

The article in this number on "The Teaching of Analytic Geometry" is the first of a series of papers on the teaching of collegiate mathematics to be published in the Monthly. The editors assume no responsibility for the propositions set forth in these papers, but they welcome the expression of views on this subject as an indication of life and a recognition of room for improvement. They feel sure that many interesting and valuable contributions can be made along this line by those whose experience in various particular phases of the work enables them to speak with some authority, and it is hoped that the papers already arranged for may call forth others, to the end that the various points of view may be fully and clearly set forth.

Renewals and the securing of new subscriptions for 1910 are now in order. In this connection, we wish to thank Professor M. E. Graber, of Heidelberg University, Tiffin, Ohio, for the interest he is taking in presenting the claims of the MONTHLY to the Teachers of Mathematics in Ohio. He did good work for the MONTHLY at the meeting of Science and Mathematics Teachers which met at Columbus, Ohio, December 29th. One such enthusiastic supporter in every State in the Union would soon put the MONTHLY on a self-sustaining basis. F.

The second meeting of the Science and Mathematics Section of the Ohio College Association was held on Wednesday morning, December 29th, in Chemistry Hall, Ohio State University, where the following papers were read:

1. The Nature and Value of a Lecture Course on the Modern Concepts of Mathematics open to Juniors and Seniors. Prof. Henry D. Coar, Marietta College.

2. The Nature of a First Course in Analytic Geometry. Prof. Grace M. Bareis, Ohio State University.

3. What should be the Range of Work in Chemistry for a Liberal Education? Prof. William B. Bentley, Ohio University.

4. What is the Cultural Value of Geology in a College Course? Prof. Lewis W. Westgate, Ohio Wesleyan University.

The officers for the year 1910 are, president, Prof. H. D. Coar; vice president, Prof. W. B. Bentley; and secretary, Prof. M. E. Graber. SEC.

☞ Title page of Vols. XV and XVI were put in May number.

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